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Estimation of location and scale parameters for the Burr XII distribution using generalized order statistics

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Abstract

The minimum variance linear unbiased estimators (MVLUE), the best linear invariant estimators (BLIE) and the maximum likelihood estimators (MLE) based on n-selected generalized order statistics are presented for the parameters of the Burr XII distribution.

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1. Introduction

We say that a random variable X has the Burr XII distribution with the parameters β, λ, μ and σ if

$$F(x) = 1 - \beta^{\lambda}\left(\beta + \frac{x-\mu}{\sigma}\right)^{-\lambda}, \quad x > \mu,\ \beta > 0,\ \lambda > 0,\ -\infty < \mu < \infty,\ \sigma > 0. \tag{1}$$

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From the Burr XII distribution we get the generalized Pareto distribution ($\lambda = \beta = \alpha^{-1}$) and the Lomax distribution ($\beta = 1$) (cf. [1]). In [1] various estimators of the scale parameter σ and the location parameter μ for various classes of distributions (Gumbel distribution, power distribution, Weibull distribution, Rayleigh distribution, logistic distribution, Pareto distribution) based on record values were given. The use of generalized order statistics to construct estimators was discussed in [2–4], for example estimators of the scale parameter σ and the location parameter μ for the exponential distribution, power distribution and the Lomax distribution. Some of those results are generalized in this paper.

We give the maximum likelihood (MLE), best linear invariant (BLIE), minimum variance unbiased (MVLUE) estimators of parameters μ and σ for the Burr XII distribution using generalized order statistics.

2. Preliminaries

We recall the concept of generalized order statistics (cf. [6]).

Let F be an absolutely continuous distribution function with density function f. Let $n \in \mathbb{N}$, $\tilde{m} = (m_1, \dots, m_{n-1}) \in \mathbb{R}^{n-1}$, $k > 0$, be given constants such that for all $1 \leqslant i \leqslant n-1$,

$$\gamma_i = k + n - i + M_i > 0,$$

where $M_i = \sum_{j=i}^{n-1} m_j$. The random variables $X(r, n, \tilde{m}, k)$, $1 \leqslant r \leqslant n$, are said to be generalized order statistics if their joint density function is of the form

$$\begin{aligned} & f^{(X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k))}(x_1, x_2, \dots, x_n) \\ & \quad = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(\overline{F}(x_i))^{m_i} f(x_i)\right)(\overline{F}(x_n))^{k-1} f(x_n) \end{aligned} \tag{2}$$

for $F^{-1}(0) < x_1 \leqslant \cdots \leqslant x_n < F^{-1}(1)$, where

$$\overline{F}(x) = 1 - F(x).$$

The marginal density function of the rth generalized order statistics for $m_1 = \cdots = m_{n-1} = m$ based on the distribution function F and density function f is given by

$$f^{X(r,n,m,k)}(x) = \frac{c_{r-1}^*}{(r-1)!}(\overline{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)),$$

where

$$c_{r-1}^* = \prod_{i=1}^{r}\gamma_i \quad \text{with } \gamma_i = k + (n-i)(m+1),$$

and

$$g_m(x) = \begin{cases} (m+1)^{-1}(1-(1-x)^{m+1}), & m \neq -1, \\ -\ln(1-x), & m = -1, \end{cases} \quad x \in (0, 1).$$

The notation of generalized order statistics provides a unified approach to some distributional and moment properties of ordered random variables. The model of generalized order statistics contains many models of ordered random variables as special cases, e.g.:

1. order statistics $X_{1:n}, \dots, X_{n:n}$ of a sample $(X_1, \dots, X_n)$ of size n from cdf F are generalized order statistics with $m_1 = \cdots = m_{n-1} = 0$ and $k = 1$;

2. record values of a sequence $\{X_n, n \geqslant 1\}$ of i.i.d. random variables are generalized order statistics with $m_1 = \cdots = m_{n-1} = -1$ and $k = 1$;
3. kth record values $Y_1^{(k)}, \ldots, Y_n^{(k)}$ of an i.i.d. sequence $\{X_n, n \geqslant 1\}$ are generalized order statistics with $m_1 = \cdots = m_{n-1} = -1$ and $k \geqslant 1$;
4. Pfeifer record values $X_{\triangle_1}^{(1)}, \ldots, X_{\triangle_n}^{(n)}$ of an array $\{X_i^{(j)}, i \geqslant 1, j \geqslant 1\}$ of independent random variables such that $X_i^{(j)}, i \geqslant 1$, are identically distributed with distribution function $F_j(x) = 1 - (1 - F(x))^{\beta_j}$, $j \geqslant 1$, where $\beta_j > 0$ for $j \geqslant 1$, are generalized order statistics with $m_i = \beta_i - \beta_{i+1} - 1$ and $k = \beta_n$, (cf. [10]);
5. progressive type II censored order statistics $X_{1:n,N}^{(\widetilde{R})}, \ldots, X_{n:n,N}^{(\widetilde{R})}$, where $\widetilde{R} = (R_1, \ldots, R_n)$ and $R_i \in \mathbb{N}_0$, $1 \leqslant i \leqslant n$, are generalized order statistics with $m_i = R_i$ and $k = R_n + 1$, (cf. [5] and references therein);
6. sequential order statistics $X_*^{(1)}, \ldots, X_*^{(n)}$ of an array $\{Y_j^{(i)}, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n - i + 1\}$ of independent random variables such that $Y_j^{(i)}, 1 \leqslant j \leqslant n - i + 1$, are identically distributed with distribution function $F_i(x) = 1 - (1 - F(x))^{\alpha_i}$, for $1 \leqslant i \leqslant n$, are generalized order statistics with $m_i = (n - i + 1)\alpha_i - (n - i)\alpha_{i+1} - 1$ and $k = \alpha_n$, (cf. [6]).

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent observations of X and

$$^*X_n = \frac{X_n - \mu}{\sigma}, \quad n = 1, 2, \ldots,$$

denote the standardized variants which may be regarded as independent observations on the standardized variate

$$^*X = \frac{X - \mu}{\sigma}.$$

Let $X(1, n, m, k), \ldots, X(n, n, m, k)$ be the first n of generalized order statistics from $\{X_n, n \geqslant 1\}$. Then

$$^*X(r, n, m, k) = \frac{X(r, n, m, k) - \mu}{\sigma}, \quad r = 1, 2, \ldots, n,$$

are the sequences of generalized order statistics from $\{^*X_n, n \geqslant 1\}$. Write

$$\begin{aligned}
&\alpha_r := \mathrm{E}[^*X(r, n, m, k)],\\
&\omega_{rr} := \mathrm{Var}[^*X(r, n, m, k)],\\
&\omega_{rs} := \mathrm{Cov}[^*X(r, n, m, k), {}^*X(s, n, m, k)],\\
&r, s = 1, 2, \ldots n; r < s.
\end{aligned}$$

Reverting now to the observations we have

$$\begin{aligned}
\mathrm{E}[X(r, n, m, k)] &= \mu + \sigma\alpha_r,\\
\mathrm{Var}[X(r, n, m, k),] &= \sigma^2\omega_{rr},\\
\mathrm{Cov}[X(r, n, m, k), X(s, n, m, k)] &= \sigma^2\omega_{rs}.
\end{aligned} \tag{3}$$

We see that the expectation are linear functions of the parameters μ and σ with known coefficients α_r and the variances and covariances are known up to a scale factor σ^2. The least-squares theorem of Gauss and Markov (cf. [11]) will be applied to obtain linear unbiased estimators μ and σ with minimal variance.

3. Estimators of parameters for the Burr XII distribution

3.1. Minimum variance linear unbiased estimators (MVLUE)

Here we consider the estimation of the location parameter μ and scale parameter σ for the Burr XII distribution when the parameters λ and β are known. We need the following:

Lemma 1. *For $1 \leqslant r < s \leqslant n$ and for $\lambda > 2/\gamma_j$, $j = 1, \dots, r$ let $X(r, n, m, k)$, $X(s, n, m, k)$ be the* rth *and the* sth *generalized order statistics from the Burr XII distribution* (1). *Write*

$$c_r = \prod_{j=1}^{r} \gamma_j \lambda, \quad d_r = \prod_{j=1}^{r} (\gamma_j \lambda - 1) \quad \textit{and} \quad e_r = \prod_{j=1}^{r} (\gamma_j \lambda - 2).$$

Then in (3)

$$\alpha_r = \beta \left(\frac{c_r}{d_r} - 1 \right), \quad \omega_{rr} = (a_r - b_r) b_r, \quad \omega_{rs} = (a_r - b_r) b_s,$$

where

$$a_r = \beta \frac{d_r}{e_r}, \quad b_r = \beta \frac{c_r}{d_r}.$$

Proof. We consider the standardized Burr XII distribution in the form:

$$^*F(x) = 1 - \beta^{\lambda} (\beta + x)^{-\lambda}, \quad x > 0,\ \beta > 0,\ \lambda > 0. \tag{4}$$

The density function is given by

$$^*f(x) = \lambda \beta^{\lambda} (\beta + x)^{-(\lambda+1)}, \quad x > 0,\ \beta > 0,\ \lambda > 0.$$

Let $^*X(1, n, m, k), \dots, {}^*X(n, n, m, k)$ be the first n generalized order statistics of $\{^*X_n, n \geqslant 1\}$ from the Burr XII distribution given by (4). Then

$$\begin{aligned} \mathrm{E}[^*X(r, n, m, k)] = {} & \int_0^{\infty} \frac{c^*_{r-1}}{\Gamma(r)} x [(\beta^{\lambda} (\beta + x)^{-\lambda})]^{\gamma_r - 1} \\ & \times \left[\frac{1}{m+1} (1 - (\beta^{\lambda} (\beta + x)^{-\lambda})^{m+1}) \right]^{r-1} \lambda \beta^{\lambda} (\beta + x)^{-(\lambda+1)} \mathrm{d}x, \end{aligned}$$

which after substitution $t = \beta^{\lambda} (\beta + x)^{-\lambda}$ gives

$$\begin{aligned} \mathrm{E}[^*X(r, n, m, k)] &= \int_0^1 \frac{c^*_{r-1}}{\Gamma(r)} (-\beta + \beta t^{-1/\lambda}) t^{\gamma_r - 1} \left[\frac{1}{m+1} (1 - t^{m+1}) \right]^{r-1} \mathrm{d}t \\ &= \frac{c^*_{r-1}}{\Gamma(r)} \frac{\beta}{(m+1)^r} B\left(r, \frac{\lambda \gamma_r - 1}{\lambda(m+1)} \right) - \frac{c^*_{r-1}}{\Gamma(r)} \frac{\beta}{(m+1)^r} B\left(r, \frac{\lambda \gamma_r}{\lambda(m+1)} \right) \\ &= \beta \left(\frac{c_r}{d_r} - 1 \right) = \alpha_r. \end{aligned}$$

Then by (3) we have

$$\mathrm{E}[X(r, n, m, k)] = \mu + \alpha_r \sigma.$$

Similarly it can be shown that

$$\mathrm{E}[{}^*X(r,n,m,k)]^2 = \beta^2\left[1 - 2\frac{c_r}{d_r} + \frac{c_r}{e_r}\right] \quad \text{for } r = 1, 2, \dots, n.$$

Thus the variance

$$\mathrm{Var}[{}^*X(r,n,m,k)] = (a_r - b_r)b_r,$$

and by (3) we obtain

$$\mathrm{Var}[X(r,n,m,k)] = \sigma^2(a_r - b_r)b_r, \quad r = 1, 2, \dots, n.$$

Now we know that for $1 \leqslant r \leqslant s-1$,

$$\begin{aligned}
&\mathrm{E}[{}^*X(r,n,m,k)\,{}^*X(s,n,m,k)] \\
&\quad = \frac{\beta}{\lambda\gamma_s - 1}\mathrm{E}[{}^*X(r,n,m,k)] + \frac{\lambda\gamma_s}{\lambda\gamma_s - 1}\mathrm{E}[{}^*X(r,n,m,k)\,{}^*X(s-1,n,m,k)] \\
&\quad = \sum_{p=r+1}^{s}\left[\frac{\beta}{\lambda\gamma_p - 1}\right]\prod_{i=p+1}^{s}\frac{\lambda\gamma_i}{\lambda\gamma_i - 1}E[{}^*X(r,n,m,k)] \\
&\qquad + \prod_{i=r+1}^{s}\frac{\lambda\gamma_i}{\lambda\gamma_i - 1}\mathrm{E}[{}^*X(r,n,m,k)]^2
\end{aligned}$$

and

$$\mathrm{E}[{}^*X(s,n,m,k)] = \prod_{i=r+1}^{s}\frac{\lambda\gamma_i}{\lambda\gamma_i - 1}\mathrm{E}[{}^*X(r,n,m,k)] + \sum_{p=r+1}^{s}\left[\frac{\beta}{\lambda\gamma_p - 1}\right]\prod_{i=p+1}^{s}\frac{\lambda\gamma_i}{\lambda\gamma_i - 1}$$

(cf. [9, Theorems 2.1 and 3.1]).

Hence

$$\begin{aligned}
\omega_{rs} &= \mathrm{Cov}[{}^*X(r,n,m,k)\,{}^*X(s,n,m,k)] \\
&= \prod_{i=r+1}^{s}\frac{\lambda\gamma_i}{\lambda\gamma_i - 1}E^2[{}^*X(r,n,m,k)] - \prod_{i=r+1}^{s}\frac{\lambda\gamma_i}{\lambda\gamma_i - 1}(E[{}^*X(r,n,m,k)])^2 \\
&= \frac{c_s}{c_r}\frac{d_r}{d_s}\mathrm{Var}[{}^*X(r,n,m,k)] = (a_r - b_r)b_s
\end{aligned}$$

and by (3)

$$\mathrm{Cov}[X(r,n,m,k)X(s,n,m,k)] = \sigma^2(a_r - b_r)b_s, \quad r, s = 1, 2, \dots, n,$$

which completes the proof of lemma. □

Theorem 1. *Let $X(1,n,m,k), \dots, X(n,n,m,k)$ be the first n generalized order statistics from the Burr XII distribution. The MVLUE of μ and σ, for known λ and β are given by*

$$\hat{\mu}_{\mathrm{GM}} = \left[\sum_{i=1}^{n} w_{1i}X(i,n,m,k)\right], \quad \hat{\sigma}_{\mathrm{GM}} = \frac{(\gamma_1\lambda - 1)}{\beta}(X(1,n,m,k) - \hat{\mu}_{\mathrm{GM}}), \tag{5}$$

where

$$w_{11} = \frac{1}{D_0}\left\{T_0 e_1(\gamma_1\lambda - 1) - \frac{e_1^2}{c_1}(\gamma_1\lambda - \gamma_2\lambda + 1)\right\},$$
$$w_{1i} = \frac{1}{D_0}\left\{-\frac{e_1 e_i}{c_i}(\lambda\gamma_i - \lambda\gamma_{i+1} + 1)\right\}, \quad i = 2, \ldots, n-1,$$
$$w_{1n} = -\frac{1}{D_0}e_1(\gamma_n\lambda - 1)\frac{e_n}{c_n}, \tag{6}$$

$$D_0 = e_1 c_1 T_0 - e_1^2, \quad T_0 = \sum_{i=1}^{n}\frac{e_i}{c_i},$$

with e_i, c_i, d_i given in lemma.

The variance and the covariance of the estimators are given by

$$\mathrm{Var}(\hat{\mu}_{\mathrm{GM}}) = \sigma^2\beta^2\frac{T_0}{D_0}, \quad \mathrm{Var}(\hat{\sigma}_{\mathrm{GM}}) = \sigma^2\frac{T_0 + e_1^2}{D_0},$$
$$\mathrm{Cov}(\hat{\mu}_{\mathrm{GM}}, \hat{\sigma}_{\mathrm{GM}}) = \sigma^2\beta\frac{T_0 - e_1}{D_0}. \tag{7}$$

Proof. Let $\boldsymbol{X}' = (X(1, n, m, k), \ldots, X(n, n, m, k))$ be the vector of generalized order statistics. Then

$$\mathrm{E}\boldsymbol{X} = \mu\mathbf{1} + \boldsymbol{\alpha}\sigma,$$

where

$$\boldsymbol{\alpha}' = (\alpha_1, \alpha_2, \ldots, \alpha_n) \quad \text{with } \alpha_r = \beta\left(\frac{c_r}{d_r} - 1\right).$$

We note that $\mathrm{E}\boldsymbol{X}$ can be written:

$$\mathrm{E}\boldsymbol{X} = \boldsymbol{p}\boldsymbol{\Theta},$$

where $\boldsymbol{p}$ is the $(n \times 2)$ matrix $(\mathbf{1}, \boldsymbol{\alpha})$ and $\boldsymbol{\Theta}' = (\mu, \sigma)$. The variance matrix of $\boldsymbol{X}$ i.e. the matrix of variances and covariances, is

$$V(\boldsymbol{X}) = \sigma^2\boldsymbol{\omega},$$

where $\boldsymbol{\omega}$ is the $(n \times n)$ symmetric positive-definite matrix of all the ω_{rs} with

$$\omega_{rs} = (a_r - b_r)b_s, \quad \text{where } a_r = \beta\frac{d_r}{e_r}, b_r = \beta\frac{c_r}{d_r}.$$

So we consider the general linear model of Gauss–Markov (cf. [11, pp. 122–123]). Using the Gauss–Markov theorem, the linear unbiased estimators of $\boldsymbol{\Theta}'$ are given by

$$\hat{\boldsymbol{\Theta}} = \begin{pmatrix}\hat{\mu}\\ \hat{\sigma}\end{pmatrix} = (\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'\boldsymbol{U} = \boldsymbol{W}\boldsymbol{X}, \tag{8}$$

where $\boldsymbol{U} := (\boldsymbol{T}')^{-1}\boldsymbol{X}, \boldsymbol{B} := (\boldsymbol{T}')^{-1}\boldsymbol{p}, \boldsymbol{T}'$ is a matrix such that $\boldsymbol{\omega} = \boldsymbol{T}'\boldsymbol{T}$ and $\boldsymbol{W} := (\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'(\boldsymbol{T}')^{-1}$ (cf. [11]), i.e.

$$\boldsymbol{W} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \end{bmatrix}. \tag{9}$$

Since $\boldsymbol{\omega}$ is positive defined then there exists an $(n \times n)$ matrix $\boldsymbol{T}'$ such that $\boldsymbol{\omega} = \boldsymbol{T}'\boldsymbol{T}$. Using Cholesky's decomposition of $\boldsymbol{\omega}$ we get

$$\boldsymbol{T}' = \begin{bmatrix} t_{11} & 0 & 0 & 0 \\ t_{12} & t_{22} & 0 & 0 \\ & & & 0 \\ t_{1n} & t_{2n} & & t_{nn} \end{bmatrix},$$

where $t_{11} = \sqrt{\omega_{11}}$, $t_{1j} = \frac{\omega_{1j}}{t_{11}}$, $j = 2, \ldots, n$, and

$$t_{ii} = \sqrt{\omega_{ii} - \sum_{p=1}^{i-1} t_{pi}^2}, \quad i = 2, 3, \ldots, n, \qquad t_{ij} = \frac{1}{t_{ii}}\left[\omega_{ij} - \sum_{p=1}^{i-1} t_{pi}t_{pj}\right], \quad j > i,$$
$$t_{ij} = 0, \quad i > j,\ i = 2, 3, \ldots, n-1.$$

Hence $(\boldsymbol{T}')^{-1}$ has the form

$$(\boldsymbol{T}')^{-1} = \begin{bmatrix} a_{11} & 0 & 0 & & 0 & 0 \\ a_{21} & a_{22} & 0 & & 0 & 0 \\ 0 & a_{32} & a_{33} & & 0 & 0 \\ & & & & & \\ & & & & & \\ 0 & 0 & 0 & & a_{nn-1} & a_{nn} \end{bmatrix}, \tag{10}$$

where

$$a_{11} = \frac{\lambda\gamma_1 - 1}{\beta}\sqrt{\frac{e_1}{c_1}}, \quad a_{ii} = \frac{\lambda\gamma_i - 1}{\beta}\sqrt{\frac{e_i}{c_i}}, \ \ a_{ii-1} = -\frac{\lambda\gamma_i}{\beta}\sqrt{\frac{e_i}{c_i}}, \ \ i = 2, 3, \ldots, n.$$

Then we have $\mathrm{E}(\boldsymbol{U}) = \boldsymbol{B}\boldsymbol{\Theta}$,

$$\boldsymbol{B}' = \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & \cdots & b_{n2} \end{bmatrix}, \tag{11}$$

where

$$b_{11} = \frac{\lambda\gamma_1 - 1}{\beta}\sqrt{\frac{e_1}{c_1}}, \quad b_{i1} = -\frac{1}{\beta}\sqrt{\frac{e_i}{c_i}}, \ \ i = 2, \ldots, n, \quad b_{i2} = \sqrt{\frac{e_i}{c_i}}, \ \ i = 1, \ldots, n.$$

Therefore

$$(\boldsymbol{B}'\boldsymbol{B})^{-1} = \frac{\beta^2}{D_0}\begin{bmatrix} T_0 & \frac{T_0 - e_1}{\beta} \\ \frac{T_0 - e_1}{\beta} & \frac{T_0 + e_1^2}{\beta^2} \end{bmatrix}. \tag{12}$$

From (10)–(12) we get the elements of $\boldsymbol{W}$ in (9) as follows:

$$w_{11} = \frac{1}{D_0}\left\{T_0 e_1(\gamma_1\lambda - 1) - \frac{e_1^2}{c_1}(\gamma_1\lambda - \gamma_2\lambda + 1)\right\},$$
$$w_{1i} = \frac{1}{D_0}\left\{-\frac{e_1 e_i}{c_i}(\lambda\gamma_i - \lambda\gamma_{i+1} + 1)\right\}, \quad i = 2, \ldots, n-1,$$

$$w_{1n} = -\frac{1}{D_0} e_1(\gamma_n\lambda - 1)\frac{e_n}{c_n},$$
$$w_{21} = \frac{(\gamma_1\lambda - 1)}{\beta} - \frac{(\gamma_1\lambda - 1)w_{11}}{\beta}, \quad w_{2i} = -\frac{(\gamma_1\lambda - 1)w_{1i}}{\beta}, \ \ i = 2, \dots, n,$$

and by (8) we get the estimators (5).
The variance and covariance of the estimators are given by

$$\mathrm{Var}(\widehat{\boldsymbol{\Theta}}) = \sigma^2(\boldsymbol{B}'\boldsymbol{B})^{-1}$$

(cf. [11, p. 124]) which by (12) proves (7). □

Remark 1. For $m_1 = \cdots = m_{n-1} = -1$ and $k \geqslant 1$ the estimators $\hat{\mu}_{\mathrm{GM}}$ and $\hat{\sigma}_{\mathrm{GM}}$ coincide with the estimators $\hat{\mu}_{\mathrm{GM}}^{(k)}$ and $\hat{\sigma}_{\mathrm{GM}}^{(k)}$ for μ and σ of the Burr XII distribution, based on the kth upper record values $y_1^{(k)}, y_2^{(k)}, \dots, y_n^{(k)}$ as follows:

$$\hat{\mu}_{\mathrm{GM}}^{(k)} = \left[\sum_{i=1}^{n} w_{1i} y_i^{(k)}\right], \quad \hat{\sigma}_{\mathrm{GM}}^{(k)} = \frac{d_1}{\beta}\left(y_1^{(k)} - \hat{\mu}_{\mathrm{GM}}^{(k)}\right), \tag{13}$$

where (see (6))

$$w_{11} = \frac{1}{D_0}\left\{T_0 e_1 d_1 - \frac{e_1^2}{c_1}\right\},$$
$$w_{1i} = \frac{1}{D_0}\left\{-\frac{e_{i+1}}{c_i}\right\}, \quad i = 2, \dots, n-1,$$
$$w_{1n} = -\frac{d_1}{D_0}\frac{e_{n+1}}{c_n},$$
$$D_0 = e_1 c_1 T_0 - e_1^2, \quad T_0 = \sum_{i=1}^{n} \frac{e_i}{c_i},$$

with

$$c_i = k^i\lambda^i, \quad d_i = (k\lambda - 1)^i \quad \text{and} \quad e_i = (k\lambda - 2)^i.$$

The variance and the covariance of the estimators are given by

$$\mathrm{Var}(\hat{\mu}_{\mathrm{GM}}^{(k)}) = \sigma^2\beta^2\frac{T_0}{D_0}, \quad \mathrm{Var}(\hat{\sigma}_{\mathrm{GM}}^{(k)}) = \sigma^2\frac{T_0 + e_1^2}{D_0},$$
$$\mathrm{Cov}(\hat{\mu}_{\mathrm{GM}}^{(k)}, \hat{\sigma}_{\mathrm{GM}}^{(k)}) = \sigma^2\beta\frac{T_0 - e_1}{D_0}$$

(cf. [7]).

Corollary 1. *If α is known, then the MVLUE for the parameters of the generalized Pareto distribution with the probability density function*

$$f(x) = \frac{1}{\sigma}\left(1 + \alpha\frac{x - \mu}{\sigma}\right)^{-(1+\alpha^{-1})}, \quad x \geqslant \mu, \ -\infty < \mu < \infty, \ \sigma > 0, \ \alpha > 0, \tag{14}$$

in terms of generalized order statistics, are given by

$$\hat{\mu}_{\mathrm{GM}} = \left[\sum_{i=1}^{n} w_{1i} X(i,n,m,k)\right], \quad \hat{\sigma}_{\mathrm{GM}} = (\gamma_1 - \alpha)(X(1,n,m,k) - \hat{\mu}_{\mathrm{GM}}), \tag{15}$$

where

$$\begin{aligned}
w_{11} &= \frac{1}{D_0}\left\{T_0 e_1\left(\frac{\gamma_1}{\alpha} - 1\right) - \frac{e_1^2}{c_1}\left(\frac{\gamma_1}{\alpha} - \frac{\gamma_2}{\alpha} + 1\right)\right\},\\
w_{1i} &= \frac{1}{D_0}\left\{-\frac{e_1 e_i}{c_i}\left(\frac{\gamma_i}{\alpha} - \frac{\gamma_{i+1}}{\alpha} + 1\right)\right\}, \quad i = 2,\dots,n-1,\\
w_{1n} &= -\frac{1}{D_0} e_1\left(\frac{\gamma_n}{\alpha} - 1\right)\frac{e_n}{c_n},\\
D_0 &= e_1 c_1 T_0 - e_1^2, \quad T_0 = \sum_{i=1}^{n} \frac{e_i}{c_i},
\end{aligned}$$

with

$$c_r = \prod_{j=1}^{r} \frac{\gamma_j}{\alpha}, \quad d_r = \prod_{j=1}^{r} \frac{\gamma_j - \alpha}{\alpha} \quad \textit{and} \quad e_r = \prod_{j=1}^{r} \frac{\gamma_j - 2\alpha}{\alpha}.$$

The variance and the covariance of the estimators are given by

$$\begin{aligned}
&\mathrm{Var}(\hat{\mu}_{\mathrm{GM}}) = \sigma^2 \alpha^{-2} \frac{T_0}{D_0}, \quad \mathrm{Var}(\hat{\sigma}_{\mathrm{GM}}) = \sigma^2 \frac{T_0 + e_1^2}{D_0},\\
&\mathrm{Cov}(\hat{\mu}_{\mathrm{GM}}, \hat{\sigma}_{\mathrm{GM}}) = \sigma^2 \alpha^{-1}\, \frac{T_0 - e_1}{D_0}.
\end{aligned}$$

We obtain these estimators from (5) *when* $\lambda = \beta = \alpha^{-1}$.

Remark 2. For $m_1 = \cdots = m_{n-1} = -1$ and $k \geqslant 1$ the estimators $\hat{\mu}_{\mathrm{GM}}$ and $\hat{\sigma}_{\mathrm{GM}}$ coincide with the estimators based on the kth upper record values $y_1^{(k)}, y_2^{(k)}, \dots, y_n^{(k)}$ (cf. [7]):

$$\hat{\mu}_{\mathrm{GM}}^{(k)} = \left[\sum_{i=1}^{n} w_{1i} y_i^{(k)}\right], \quad \hat{\sigma}_{\mathrm{GM}}^{(k)} = (k - \alpha)\left(y_1^{(k)} - \hat{\mu}_{\mathrm{GM}}^{(k)}\right), \tag{16}$$

where

$$\begin{aligned}
w_{11} &= \frac{1}{D_0}\left\{T_0 \frac{(k-2\alpha)(k-\alpha)}{\alpha^2} - \frac{(k-2\alpha)^2}{k\alpha}\right\},\\
w_{1i} &= -\frac{1}{D_0}\frac{(k-2\alpha)^{i+1}}{k^i \alpha}, \quad i = 2,\dots,n-1,\\
w_{1n} &= -\frac{1}{D_0}\frac{(k-\alpha)}{k^n \alpha^2}(k-2\alpha)^{n+1},\\
D_0 &= \frac{k-2\alpha}{\alpha^2}[kT_0 - k + 2\alpha], \quad T_0 = \sum_{i=1}^{n}\left(\frac{k-2\alpha}{k}\right)^i.
\end{aligned}$$

The variance and the covariance of the estimators are given by

$$\mathrm{Var}(\hat{\mu}_{\mathrm{GM}}^{(k)}) = \sigma^2\alpha^{-2}\frac{T_0}{D_0}, \quad \mathrm{Var}(\hat{\sigma}_{\mathrm{GM}}^{(k)}) = \sigma^2\frac{T_0 + \left(\frac{k-2\alpha}{\alpha}\right)^2}{D_0},$$

$$\mathrm{Cov}(\hat{\mu}_{\mathrm{GM}}^{(k)}, \hat{\sigma}_{\mathrm{GM}}^{(k)}) = \sigma^2\alpha^{-1}\frac{T_0 - \left(\frac{k-2\alpha}{\alpha}\right)}{D_0}.$$

Corollary 2. *If λ is known, then the MVLUE for the parameters of the Lomax distribution with the probability density function*

$$f(x) = \frac{\lambda}{\sigma}\left(1 + \frac{x-\mu}{\sigma}\right)^{-(\lambda+1)}, \quad x \geqslant \mu,\ -\infty < \mu < \infty,\ \sigma > 0,\ \lambda > 0, \tag{17}$$

in terms of generalized order statistics (cf. [3]*) are given in Theorem* 1 *when $\beta = 1$.*

Remark 3. The estimators $\hat{\mu}_{\mathrm{GM}}$ and $\hat{\sigma}_{\mathrm{GM}}$ for $m_1 = \cdots = m_{n-1} = -1$ and $k \geqslant 1$ coincide with the estimators based on the kth upper record values $y_1^{(k)}, y_2^{(k)}, \ldots, y_n^{(k)}$ (cf. [7]). They are given in Remark 1 with $\beta = 1$.

3.2. Best linear invariant estimators (BLIE)

We consider now the best invariant estimators for parameters of Burr XII distribution when β and λ are known. "Best" is used in the sense of minimum mean squared error and "invariant" with respect to location parameter.

Theorem 2. *The BLIE $\tilde{\sigma}_{\mathrm{BL}}$ and $\tilde{\mu}_{\mathrm{BL}}$ of the location and scale parameters of the Burr XII distribution* (1) *based on the first n generalized order statistics are*:

$$\tilde{\mu}_{\mathrm{BL}} = \hat{\mu}_{\mathrm{GM}} - \hat{\sigma}_{\mathrm{GM}}\left[\beta\frac{T_0 - e_1}{T_0 + D_0 + e_1^2}\right], \quad \tilde{\sigma}_{\mathrm{BL}} = \hat{\sigma}_{\mathrm{GM}}\frac{D_0}{T_0 + D_0 + e_1^2}.$$

The mean squared errors of $\tilde{\mu}_{\mathrm{BL}}$ and $\tilde{\sigma}_{\mathrm{BL}}$ are given by

$$\mathrm{MSE}(\tilde{\mu}_{\mathrm{BL}}) = \sigma^2\beta^2\left[\frac{T_0}{D_0} - \frac{(T_0 - e_1)^2}{D_0(D_0 + T_0 + e_1^2)}\right], \quad \mathrm{MSE}(\tilde{\sigma}_{\mathrm{BL}}) = \sigma^2\frac{T_0 + e_1^2}{D_0 + T_0 + e_1^2},$$

where $\hat{\sigma}_{\mathrm{GM}}$ and $\hat{\mu}_{\mathrm{GM}}$ are the MVLUE for σ and μ given by (5).

Proof. Using the method of Mann (cf. [8]) we obtain the BLIE $\tilde{\mu}_{\mathrm{BL}}$ and $\tilde{\sigma}_{\mathrm{BL}}$ for μ and σ in the form:

$$\tilde{\mu}_{\mathrm{BL}} = \hat{\mu}_{\mathrm{GM}} - \hat{\sigma}_{\mathrm{GM}}[\mathrm{E}_{12}(1 + \mathrm{E}_{22})^{-1}], \quad \tilde{\sigma}_{\mathrm{BL}} = \hat{\sigma}_{\mathrm{GM}}(1 + \mathrm{E}_{22})^{-1},$$

where E_{11}, E_{12} and E_{22} are taken from

$$\sigma^2\begin{bmatrix}\mathrm{E}_{11} & \mathrm{E}_{12}\\ \mathrm{E}_{12} & \mathrm{E}_{22}\end{bmatrix} := \begin{bmatrix}\mathrm{Var}(\hat{\mu}_{\mathrm{GM}}) & \mathrm{Cov}(\hat{\mu}_{\mathrm{GM}}, \hat{\sigma}_{\mathrm{GM}})\\ \mathrm{Cov}(\hat{\mu}_{\mathrm{GM}}, \hat{\sigma}_{\mathrm{GM}}) & \mathrm{Var}(\hat{\sigma}_{\mathrm{GM}})\end{bmatrix},$$

after using (7), i.e.

$$\mathrm{E}_{11}=\beta^2\frac{T_0}{D_0},\quad \mathrm{E}_{12}=\beta\frac{T_0-e_1}{D_0},\quad \mathrm{E}_{22}=\frac{T_0+e_1^2}{D_0}.$$

Moreover, we have

$$\mathrm{MSE}(\tilde{\mu}_{\mathrm{BL}})=\sigma^2\left[\mathrm{E}_{11}-\mathrm{E}_{12}^2(1+\mathrm{E}_{22})^{-1}\right],\quad \mathrm{MSE}(\tilde{\sigma}_{\mathrm{BL}})=\sigma^2\mathrm{E}_{22}(1+\mathrm{E}_{22})^{-1}. \qquad \square$$

Remark 4. The estimators $\tilde{\mu}_{\mathrm{BL}}$ and $\tilde{\sigma}_{\mathrm{BL}}$ for $m_1=\cdots=m_{n-1}=-1$ and $k\geqslant 1$ coincide with the estimators $\tilde{\sigma}_{\mathrm{BL}}^{(k)}$ and $\tilde{\mu}_{\mathrm{BL}}^{(k)}$ of parameters of the Burr XII distribution (1) based on the first n kth upper record values:

$$\tilde{\mu}_{\mathrm{BL}}^{(k)}=\hat{\mu}_{\mathrm{GM}}^{(k)}-\hat{\sigma}_{\mathrm{GM}}^{(k)}\left[\beta\frac{T_0-e_1}{T_0+D_0+e_1^2}\right],\quad \tilde{\sigma}_{\mathrm{BL}}^{(k)}=\hat{\sigma}_{\mathrm{GM}}^{(k)}\frac{D_0}{T_0+D_0+e_1^2},$$

and the mean squared errors of $\tilde{\mu}_{\mathrm{BL}}^{(k)}$ and $\tilde{\sigma}_{\mathrm{BL}}^{(k)}$ are given by

$$\mathrm{MSE}(\tilde{\mu}_{\mathrm{BL}}^{(k)})=\sigma^2\beta^2\left[\frac{T_0}{D_0}-\frac{(T_0-e_1)^2}{D_0(D_0+T_0+e_1^2)}\right],\quad \mathrm{MSE}(\tilde{\sigma}_{\mathrm{BL}}^{(k)})=\sigma^2\frac{T_0+e_1^2}{D_0+T_0+e_1^2},$$

where $\hat{\sigma}_{\mathrm{GM}}^{(k)}$ and $\hat{\mu}_{\mathrm{GM}}^{(k)}$ are the MVLUE for σ and μ given by (13).

Corollary 3. *If α is known, then the BLIE for the parameters of the generalized Pareto distribution given by* (14) *in terms generalized order statistics have the following form*:

$$\tilde{\mu}_{\mathrm{BL}}=\hat{\mu}_{\mathrm{GM}}-\hat{\sigma}_{\mathrm{GM}}\frac{1}{\alpha}\left[\frac{T_0-e_1}{T_0+D_0+e_1^2}\right],\quad \tilde{\sigma}_{\mathrm{BL}}=\hat{\sigma}_{\mathrm{GM}}\frac{D_0}{T_0+D_0+e_1^2},$$

where $\hat{\sigma}_{\mathrm{GM}}$ and $\hat{\mu}_{\mathrm{GM}}$ are the MVLUE for σ and μ given by (15). *The mean squared errors of $\tilde{\mu}_{\mathrm{BL}}$ and $\tilde{\sigma}_{\mathrm{BL}}$ are given by*

$$\mathrm{MSE}(\tilde{\mu}_{\mathrm{BL}})=\sigma^2\frac{1}{\alpha^2}\left[\frac{T_0}{D_0}-\frac{(T_0-e_1)^2}{D_0(D_0+T_0+e_1^2)}\right],\quad \mathrm{MSE}(\tilde{\sigma}_{\mathrm{BL}})=\sigma^2\frac{T_0+e_1^2}{D_0+T_0+e_1^2}.$$

Remark 5. The estimators $\tilde{\mu}_{\mathrm{BL}}$ and $\tilde{\sigma}_{\mathrm{BL}}$ for $m_1=\cdots=m_{n-1}=-1$ and $k\geqslant 1$ coincide with the estimators $\tilde{\sigma}_{\mathrm{BL}}^{(k)}$ and $\tilde{\mu}_{\mathrm{BL}}^{(k)}$ of parameters of the generalized Pareto distribution given by (14) based on the first n kth upper record values (cf. [7]):

$$\tilde{\mu}_{\mathrm{BL}}^{(k)}=\hat{\mu}_{\mathrm{GM}}^{(k)}-\hat{\sigma}_{\mathrm{GM}}^{(k)}\frac{1}{\alpha}\left[\frac{T_0-\left(\frac{k-2\alpha}{\alpha}\right)}{T_0+D_0+\left(\frac{k-2\alpha}{\alpha}\right)^2}\right],\quad \tilde{\sigma}_{\mathrm{BL}}^{(k)}=\hat{\sigma}_{\mathrm{GM}}^{(k)}\frac{D_0}{T_0+D_0+\left(\frac{k-2\alpha}{\alpha}\right)^2},$$

where $\hat{\sigma}_{\mathrm{GM}}^{(k)}$ and $\hat{\mu}_{\mathrm{GM}}^{(k)}$ are the MVLUE for σ and μ given by (16). The mean squared errors of $\tilde{\mu}_{\mathrm{BL}}^{(k)}$ and $\tilde{\sigma}_{\mathrm{BL}}^{(k)}$ are given by

$$\mathrm{MSE}(\tilde{\mu}_{\mathrm{BL}}^{(k)})=\sigma^2\frac{1}{\alpha^2}\left[\frac{T_0}{D_0}-\frac{\left(T_0-\left(\frac{k-2\alpha}{\alpha}\right)\right)^2}{D_0\left(D_0+T_0+\left(\frac{k-2\alpha}{\alpha}\right)^2\right)}\right],$$

$$\mathrm{MSE}(\tilde{\sigma}_{\mathrm{BL}}^{(k)})=\sigma^2\frac{T_0+\left(\frac{k-2\alpha}{\alpha}\right)^2}{D_0+T_0+\left(\frac{k-2\alpha}{\alpha}\right)^2}.$$

Corollary 4. *If λ is known, then the BLIE for the parameters of the Lomax distribution given by* (17) *in terms generalized order statistics are given in Theorem* 2 *when $\beta = 1$.*

Remark 6. The estimators $\tilde{\mu}_{\mathrm{BL}}$ and $\tilde{\sigma}_{\mathrm{BL}}$ for $m_1 = \cdots = m_{n-1} = -1$ and $k \geqslant 1$ coincide with the estimators based on the kth upper record values $y_1^{(k)}, y_2^{(k)}, \ldots, y_n^{(k)}$ (cf. [7]). They are given in Remark 4 with $\beta = 1$.

3.3. Maximum likelihood estimators (MLE)

The likelihood function L based on the first n generalized order statistics for the Burr XII distribution has the form:

$$\begin{aligned} L(\mu, \sigma | \underline{X}) &= k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} \left(\overline{F}(x_i) \right)^{m_i} f(x_i) \right) \left(\overline{F}(x_n) \right)^{k-1} f(x_n) \\ &= k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left[\beta^{\lambda} \left(\beta + \frac{x_n - \mu}{\sigma} \right)^{-\lambda} \right]^{k} \left[\frac{\lambda}{\sigma} \left(\beta + \frac{x_n - \mu}{\sigma} \right)^{-1} \right] \\ &\quad \times \left(\prod_{i=1}^{n-1} \left[\beta^{\lambda} \left(\beta + \frac{x_i - \mu}{\sigma} \right)^{-\lambda} \right]^{m_i+1} \left[\frac{\lambda}{\sigma} \left(\beta + \frac{x_i - \mu}{\sigma} \right)^{-1} \right] \right) \quad \text{(see (2))}, \end{aligned}$$

where

$$\underline{X} = (X(1, n, m, k), \ldots, X(n, n, m, k)).$$

Hence

$$\begin{aligned} \ln L(\mu, \sigma, \beta | \underline{X}) &= \ln k + \ln \left(\prod_{j=1}^{n-1} \gamma_j \right) + n \ln \lambda - n \ln \sigma \\ &\quad + \sum_{i=1}^{n-1} \left\{ (m_i + 1) \ln \left[\beta^{\lambda} \left(\beta + \frac{x_i - \mu}{\sigma} \right)^{-\lambda} \right] - \ln \left(\beta + \frac{x_i - \mu}{\sigma} \right) \right\} \\ &\quad + k \ln \left[\beta^{\lambda} \left(\beta + \frac{x_n - \mu}{\sigma} \right)^{-\lambda} \right] - \ln \left(\beta + \frac{x_n - \mu}{\sigma} \right). \end{aligned} \tag{18}$$

Differentiating (18) with respect to σ and μ we lead to

$$\begin{aligned} &-\frac{n}{\sigma} + \sum_{i=1}^{n-1} \frac{((m_i + 1)\lambda + 1)(x_i - \mu)}{\sigma^2 \left(\beta + \frac{x_i - \mu}{\sigma} \right)} + \frac{(k\lambda + 1)(x_n - \mu)}{\sigma^2 \left(\beta + \frac{x_n - \mu}{\sigma} \right)} = 0, \\ &\sum_{i=1}^{n-1} \frac{(m_i + 1)\lambda + 1}{\sigma \left(\beta + \frac{x_i - \mu}{\sigma} \right)} + \frac{k\lambda + 1}{\sigma \left(\beta + \frac{x_n - \mu}{\sigma} \right)} = 0. \end{aligned}$$

When λ and β are known the MLE of μ and σ can be obtained by numerical solution of these equations.

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